A PERIODIC REPARABLE-ITEM INVENTORY MODEL

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by

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ABSTRACT

The purpose of this thesis is to structure a periodic review model for a reparable item system experiencing normally distributed random demand. The items are issued upon demand, repaired (if possible) after use or failure, and subsequently reissued. The model addresses the requirement of system replenishment as items are lost or determined to be beyond economical repair. The inventory of ready-for-issue items is treated as two separate inventories, i. e., an inventory of those items received directly from the manufacturer and an inventory of those items received from the repair facility. Annual system cost expressions are developed as a function of the desired protection level and length of review period.

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TABLE OF SYMBOLS

R	-	Repair order leadtime (periods).
L	-	Purchase order leadtime (periods).
X_i	-	Random variable representing system demand during period i.
σ_X^2	-	Variance of X.
μ_X	-	Mean of X.
$I_2(i)$	-	Random variable representing repaired item inventory level at inventory control point (ICP) at the end of period i.
σ_2^2	-	Variance of I_2.
μ_2	-	Mean of I_2.
$I_1(i)$	-	Random variable representing ICP purchased item inventory level at the end of period i.
σ_1^2	-	Variance of I_1.
μ_1	-	Mean of I_1.
V	-	Random variable representing non-ready-for-issue (NRFI) inventory level at the repair activity.
σ_V^2	-	Variance of V.
μ_V	-	Mean of V.
$I(i)$	-	Random variable representing total inventory level $(I_1 + I_2)$ under the control of ICP at the end of period i.
σ^2	-	Variance of I.
μ	-	Mean of I.
r	-	Fixed recovery rate (percentage).
α	-	Protection level.
Q_i	-	Quantity requested by ICP repair order submitted at the beginning of period i.

q_i - Batch quantity that the repair facility started repairing at the beginning of period i.

P_i - Quantity requested by ICP purchase order submitted at the beginning of period i.

K^1 - Average number of periods to release full order at repair facility.

K - Turn around time.

A_P - Fixed procurement order cost (per order).

A_R - Fixed repair order cost (per order).

h - Ready-for-issue (RFI) inventory holding cost per item per period at ICP.

h_V - NRFI inventory holding cost per item per period at repair activity.

H_1 - ICP high limit for the purchased-item system.

H_2 - ICP high limit for the repaired-item system.

T - Basic model's period length (in year units).

1. INTRODUCTION

An existing reparable inventory system within the Naval Aviation Supply System is an exceedingly complex probabilistic system involving tens of thousands of line items and a network of stock points and repair points. Anyone attempting to model a system of this complexity is immediately aware that the model can never be more than a partial representation of reality. Despite this awareness of incompleteness, the fact that models have proven to be invaluable aids in managing complex systems stands as a primary motivating force behind the efforts to model a reparable inventory system.

The models developed by Hoekstra [1], McNall and Hatchett [2], Schrady [3], and Allen and D'Esopo [4] attest to the recent efforts to model this system. The purpose of this paper is to add to this growing knowledge base by structuring a periodic review model for a reparable item system experiencing normally distributed random demands. The model, in turn, provides a means of understanding the parameter interactions and cost-protection trade-offs that exist within a system of this nature. The periodic model techniques used by Hanssman [5] in the development of a multilevel production control model are relied upon during the initial phase of the model development.

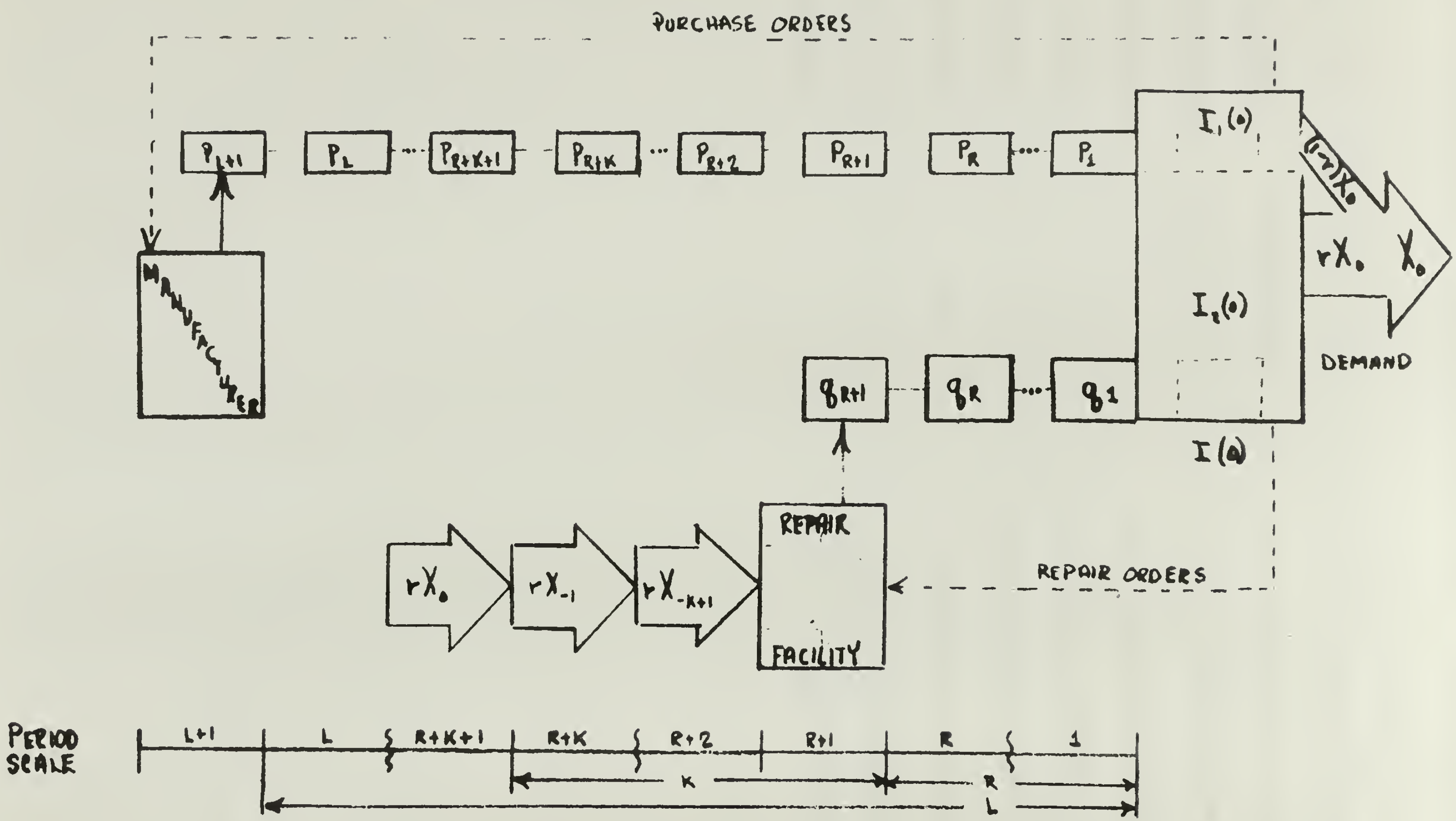
In addition to the model's basic assumptions of periodic review and normal demand, it was necessary to further simplify the reparable item system by making additional assumptions. The major assumptions are included in the following summary of the scope of the model.

The model addresses the reparable item system on a single item basis. The system is considered to be made up of two subsystems: (1) the repaired item subsystem, and (2) the purchased (new) item subsystem. The purchased item subsystem is made up of a single manufacturer, the outstanding purchase order quantities, and the purchased item portion of the ready-for-issue (RFI) inventory under the control of the inventory control point (ICP). The repaired item subsystem is made up of a single repair facility (with an inventory of non-ready-for-issue (NRFI) items), outstanding repair order quantities, and the repaired item portion of the RFI inventory under the control of the same ICP. The total demand each period is apportioned into two segments based on the recovery rate r; i.e., r percent of the demand is charged against the ICP repaired item inventory and the remaining portion is satisfied from the ICP purchased item inventory. The assumption is made that the recoverable portion of the items demanded each period will be returned to the repair facility in a given number of periods after the period in which they were demanded.

The basic model assumes that repair and purchase orders are initiated each period by the ICP. A sufficient quantity is ordered each period to bring the subsystems' inventory positions up to established limits. Separate net inventory distributions are developed for the ICP repaired item inventory, ICP purchase order inventory, ICP combined purchased and repaired item inventory, and the repair facility inventory. The relationship between the high limits of the subsystem and the

corresponding protection level, i. e., one minus the probability of stockout, is determined and corresponding system-cost equations are derived.

The equations of the basic model are then slightly modified to allow for extension of the repair and purchase review periods in multiples of the basic review period. The extension does not require that these two review periods be of equal length, as was assumed in the basic model. The annual operating cost trade-off that exists between (1) period review and order costs and (2) ICP inventory holding costs for a given protection level is examined. A method for determining the operating procedures that minimize the expected annual operating cost, within the evaluation restrictions of the extended model, are then investigated.



Flow Diagram of Basic Periodic Review Model

FIGURE 1

2. A PERIODIC REPARABLE MODEL

2.1 Description of Reparable Item System

Figure 1 illustrates the periodic structure and relative leadtime relationships of the reparable item system addressed and modeled by this paper.

The description of the system will commence at the ICP. The inventory, I, under the control of the ICP is looked upon as two inventories; i.e., those items received directly from the manufacturer, I_1, and those items received from the repair facility, I_2. The total withdrawal quantity (demand) during period i is a random variable denoted by X_i. It is assumed that period demands, $X_1, X_2, \ldots,$ are independent and identically distributed as a normal random variable X having mean μ_X and variance σ_X^2. The recovery rate, r, is a known percentage; that is, r percent of the items demanded each period will be returned in a reparable state to the repair facility. Each period rX_i of the total items demanded are satisfied from I_2 and the remaining portion, $(1 - r)X_i$, of the total demand is satisfied from I_1.

At the end of period zero, the combined ICP inventory position, I(0), consists of $I_1(0)$ plus $I_2(0)$. The outstanding orders at this moment in time are shown in Figure 1. The repair order quantity q_i and the purchase order quantity P_i go into ICP inventory at the beginning of period i. The purchase order leadtime is L periods and the repair order leadtime is R periods, where leadtime is defined as the fixed interval of periods between the time the ICP decides to place an order

and the time the ordered quantity is placed in the ICP controlled inventory. Just prior to the beginning of period 1, the new orders, Q_{R+1} and P_{L+1}, must be determined.

It is assumed that at the beginning of each period the repair facility receives the reparable portion of the system demand experienced by the system K periods earlier, where K is defined as the time in periods from when an item is demanded and the replaced item is returned to the repair facility. An ICP repair order Q is received by the repair facility at the beginning of each period. Upon receipt of the order, the repair facility immediately commences a batch repair to fill the order with the existing stock on hand. If there is insufficient stock on hand to meet the order at this point in time, the unfilled portion of the new order is included in the batch repair initiated at the beginning of the next period.

2.2 ICP Purchased Inventory Distribution and Order Rules

At the end of each period, a new purchase order, P_{L+1}, is determined by the ICP and transmitted to the manufacturer. It is assumed that the manufacturer has sufficient raw material on hand to commence production of the ordered items immediately upon receipt of the ICP order. Therefore, the first I_1 inventory level that the ICP can control is $I_1(L+1)$, where

$$I_1(L+1) = I_1(0) + P_1 + P_2 + \dots + P_L + P_{L+1}$$

$$- ((1-r)X_1 + \dots + (1-r)X_{L+1}) \quad . \qquad (1)$$

Let H_1 be defined as the high limit of the purchased item system, which is a fixed value to be determined by management. This limit represents the maximum level of items in the purchased item system, i. e., ICP purchased item inventory and outstanding purchase order quantities. Each period an order is placed to bring the purchased item system back up to this level. It follows that

$$H_1 = I_1(0) + \sum_{i=1}^{L} P_i + P_{L+1} \tag{2}$$

or

$$P_{L+1} = \text{new order} = H_1 - I_1(0) - \sum_{i=1}^{L} P_i \quad .$$

Based on equation (2), we can write (1) as follows:

$$I_1(L+1) = H_1 - \sum_{i=1}^{L+1} (1-r) X_i \quad . \tag{3}$$

As shown in Appendix A, I_1 is a normally distributed random variable with parameters

$$\mu_1 = H_1 - (L+1)(1-r)\mu_X \tag{4}$$

and

$$\sigma_1^2 = (L+1)(1-r)^2 \sigma_X^2 \quad , \tag{5}$$

where a negative inventory is considered to be an outstanding backorder quantity when the on-hand inventory level is zero.

The value of the mean parameter μ_1 can also be written as a function of a desired inventory protection level. Given a desired protection

level α , the corresponding probability of stockout is $1 - \alpha$, which can be written

$$P[I_1 \leq 0] = 1 - \alpha \quad .$$

This, in turn, can be written in the standard normal distribution function form, $\Phi(i)$, as follows:

$$\Phi\left(\frac{0 - \mu_1}{\sigma_1}\right) = 1 - \alpha$$

or

$$\Phi\left(\frac{\mu_1}{\sigma_1}\right) = \alpha \quad .$$

Defining $N(\alpha)$ as the value obtained from a standard normal distribution table corresponding to the area α under the curve,

$$N(\alpha) = \frac{\mu_1}{\sigma_1} \quad . \tag{6}$$

Rearranging and replacing σ_1 with its equivalent form shown in (5), this can be written

$$\mu_1 = N(\alpha)(L+1)^{1/2}(1-r)\sigma_X \quad . \tag{7}$$

From equations (7) and (4), the value of the high limit of the purchase system H_1 can be expressed as a function of the desired protection level α and the parameters μ_X and σ_X .

$$H_1 = (1-r)\left(N(\alpha)(L+1)^{1/2}\sigma_X + (L+1)\mu_X\right) \quad . \tag{8}$$

2.3 ICP Repaired Inventory Distribution and Order Rules

The repair facility may not be able to start on a full ICP order due to an insufficient NRFI inventory of reparable carcasses on hand. The point in time when a new repair order, Q_{R+1}, will affect the inventory at the ordering activity depends upon the leadtime and the amount of shortage at the repair facility receiving the order. As pointed out by Hanssman [5], this shortage is characterized by the average time K^1 it takes the repair facility to release the full amount of the order quantity Q_{R+1}. Referring to Figure 1, the actual output by the repair facility into q_{R+1} will be equal to the total unfilled orders or the total supply on hand, whichever is smaller. It is assumed that K^1 is an integer multiple of the unit period. Therefore, the first I_2 inventory level that the ICP can control is

$$I_2(R + K^1 + 1) \ .$$

Now,

$$I_2(R + K^1 + 1) = I_2(0) + Q_1 + Q_2 + \cdots + Q_R + Q_{R+1}$$
$$- (rX_1 + \cdots + rX_{R+K^1+1}) \ . \quad (9)$$

Let H_2 be defined as the high limit of the repaired item system which is a fixed value to be determined by management. This limit is the maximum level of items represented by the ICP repaired item inventory and outstanding repair order quantities. (The repair facility inventory is not considered within this limit.) Each period an order is placed to bring the repaired item system back up to this level. It follows that

$$H_2 = I_2(0) + \sum_{i=1}^{R} Q_i + Q_{R+1}$$

or (10)

$$Q_{R+1} = \text{new order} = H_2 - I_2(0) - \sum_{i=1}^{R} Q_i \quad .$$

Based on (10), we can write (9) as follows:

$$I_2(R + K^1 + 1) = H_2 - \sum_{i=1}^{R+K^1+1} r X_i \quad . \tag{11}$$

From (9), it can be shown (similar to the Appendix A derivation) that I_2 is a normally distributed random variable with parameters

$$\mu_2 = H_2 - (R + K^1 + 1) r \mu_X \tag{12}$$

and

$$\sigma_2^2 = (R + K^1 + 1) r^2 \sigma_X^2 \quad , \tag{13}$$

where a negative inventory is considered to be the outstanding backorder quantity when the on-hand inventory level is zero.

By the method shown in the equation (7) derivation, the value of the mean parameter μ_2 can be written as a function of a desired protection level α as follows:

$$\mu_2 = N(\alpha)\,(R + K^1 + 1)^{1/2}\, r \sigma_X \quad . \tag{14}$$

From equations (12) and (14), the value of the high limit of the repair system H_2 can be expressed as a function of the desired protection

level α and the parameters μ_X and σ_X .

$$H_2 = r\left[N(\alpha)\,(R + K^1 + 1)^{1/2}\,\sigma_X + (R + K^1 + 1)\,\mu_X \right] . \tag{15}$$

2.4 ICP Combined Inventory Distribution

Having determined the distributions of I_1 and I_2 , the parameters of the combined inventory distribution, $I = I_1 + I_2$, can now be determined. Due to the common demand distribution from which I_1 and I_2 were derived, a dependent relationship exists between I_1 and I_2 . Under these conditions, the parameters of the combined inventory distribution, I, are defined [6] as follows:

$$\mu = \mu_1 + \mu_2 \tag{16}$$

and

$$\sigma^2 = \sigma_1^2 + \sigma_2^2 + 2\,\mathrm{COV}\,(I_1 , I_2) . \tag{17}$$

Under the assumption that the joint distribution is normal, it follows that the combined inventory distribution, I, is also normally distributed with the parameters as shown above. An alternative form of the mean parameter can be obtained from (7) and (14) and the assumption that the same protection level α will be set for both the I_1 and I_2 inventories.

$$\mu = N(\alpha)\,\sigma_X\,\left((L + 1)^{1/2}(1 - r) + (R + K^1 + 1)^{1/2} r \right) . \tag{18}$$

From (5), (13), and the derivation of COV (I_1 , I_2) in Appendix B, the variance parameter can be expressed as follows:

$$\sigma^2 = \left((L + 1)(1 - r)^2 + (R + K^1 + 1) r^2 + 2r(1 - r)(R + K^1 + 1) \right) \sigma_X^2 \quad . \tag{19}$$

The above variance equation only holds if it is assumed that leadtime relationships satisfy

$$L \geq R + K^1 \quad .$$

Further, although this equation gives insight as to the effect of the various system parameters upon the combined inventory variance, it cannot be used to develop a single protection level relationship unless the separate I_1 and I_2 inventory assumption is relaxed. To relate a protection level to this term (vice separately for I_1 and I_2) would imply that material from one inventory could satisfy demand being experienced by the other inventory. In terms of practical application of the model, this interchange would be a highly desirable extension; it will be discussed in subsection 2.5.

2.5 Repair Facility Inventory Distribution

Similar to the I_1 demand derivation in Appendix A, it can be shown that the random demand, rX, charged against the I_2 inventory in a given period is normally distributed with parameters $(r\mu_X, r^2\sigma_X^2)$. Based on the repair order rules developed in subsection 2.2, the corresponding ICP repair order determined at the end of each period is equivalent to the repaired item demand during that period. It follows that the ICP repair orders can be thought of as being generated from the I_2 demand distribution given above. Further, the input into the

repair facility each period (number of reparable carcasses returned) was assumed to be generated from the same demand distribution K periods earlier. Assuming that K is greater than one period, the repair facility's net inventory position each period is a random variable determined by the difference between these two independent (K periods apart) normal random variables. Given these conditions, it follows that the repair facility's inventory V is normally distributed with parameters defined as follows:

$$\mu_V = 0 \tag{20}$$

and

$$\sigma_V^2 = 2 r^2 \sigma_X^2 \quad , \tag{21}$$

where the net value of the inventory takes on negative values (backlog) when the ICP repair order exceeds the on-hand physical inventory.

In subsequent sections where inventory carrying costs are considered, it becomes necessary to determine the expected overage so that the average holding cost of the on-hand physical inventory can be estimated. This value is found in the following manner.

Defining the overage as

$$\text{overage} = \begin{cases} V & \text{if} \quad V \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad ,$$

the expected overage is

$$E[\text{overage}] = \frac{1}{\sqrt{2\pi}\ \sigma_V} \int_0^{\infty} V \exp\left(-\frac{V^2}{2\sigma_V^2}\right) dV$$

$$= \frac{\sigma_V}{\sqrt{2\pi}} = \frac{r\sigma_X}{\sqrt{\pi}} \quad . \tag{22}$$

The expected shortage value can be determined in a similar manner. The average number of periods, K^1, it takes the repair facility to release the full amount of a repair order can be determined from the expected shortage value.

Defining the shortage as

$$\text{shortage} = \begin{cases} -V & \text{if } V \leq 0 \\ 0 & \text{otherwise} \end{cases} ,$$

the expected shortage is

$$E[\text{shortage}] = -\frac{1}{\sqrt{2\pi}\ \sigma_V} \int_{-\infty}^{0} V \exp\left(-\frac{V^2}{2\sigma_V^2}\right) dV$$

$$= \frac{\sigma_V}{\sqrt{2\pi}} = \frac{r\sigma_X}{\sqrt{\pi}} \quad . \tag{23}$$

Since $r\mu_X$ is the mean input into the repair facility each period, the expected number of periods required to accumulate sufficient material to fill the expected shortage can be represented approximately by

$$\frac{E[\text{shortage}]}{r\mu_X} = \frac{\sigma_X}{\sqrt{\pi}\ \mu_X} \quad .$$

Obviously, the preceding quotient will include fractions of periods. However, in keeping with the subsection 2.3 derivation, it becomes necessary to treat a fraction of a period as an additional whole period. This represents a conservative approach in opposition to the alternative of dropping the fraction, i.e., K^1 will be a fraction of a period larger than the expected delay time vice a fraction smaller. Accordingly, K^1 will be defined as follows:

$$K^1 = \left[\frac{\sigma_X}{\sqrt{\pi} \quad \mu_X} \right] + 1 \quad , \tag{24}$$

where [ ] is defined as the greatest integer in the enclosed value.

2.6 Operating Cost Considerations

As pointed out in [2], there are numerous costs associated with the management of a reparable inventory system. Since the construction of a mathematical model of an inventory system is motivated by a desire to improve the operating rules for controlling a particular inventory system, the pertinent costs are those which are influenced by the operating doctrine. It follows that costs that are independent of the operating doctrine need not be included in the analysis. The periodic review costs considered herein have been so restricted and, wherever possible, have been grouped under one symbol to reduce the model's notation complexity. The costs are defined as follows:

(a) Purchase Order Cost (A_P). The fixed costs associated with (1) reviewing the purchase order records and (2) preparing a purchase order.

(b) Repair Order Cost (A_R). The fixed costs associated with (1) reviewing the repair order records, (2) preparing a repair order, and (3) initiating batch repair action at the repair facility in response to a repair order (regardless of size).

(c) ICP Holding Cost (h). The cost per unit per basic review period to hold a RFI item in the ICP inventory. This cost is normally a function of item purchase cost and an estimated holding rate.

(d) Repair Holding Cost (h_V). The cost per unit per basic review period to hold a NRFI item at the repair facility. This cost is normally a function of the NRFI item value and an estimated holding rate.

Since the net inventory distributions are subsequently used to determine the value of the holding costs, it is emphasized that these distributions were derived by treating the random demand each period in a discrete manner; i.e., as if the total period demand were received at a point in time within the period. Consequently, a holding cost is incurred only if the period's on-hand inventory plus receipts exceed period demand. The period holding cost is then proportional to the excess stock on hand.

A specific shortage cost has not been included in the analysis; however, a shortage cost is implied by the decision-maker's choice of protection level. In this regard, it is envisioned that management would be presented with yearly cost estimates for various protection levels and

that the final protection level decision would be influenced by the associated costs. Accordingly, the primary objective of deriving a cost expression for the basic model is to provide a means of relating different protection levels to the expected annual operating costs corresponding to these levels of protection.

The analysis will be based on the periodic structure of the basic model presented in subsection 2.1, where the purchase and repair orders are initiated at the end of each period. Defining T as the basic period length in year units, the average annual purchase order cost is $\frac{A_P}{T}$. Similarly, the average annual repair order cost is $\frac{A_R}{T}$.

The expected period cost of holding inventory at the ICP is found by multiplying the ICP holding cost (h) times the expected value of the net inventory at the ICP. Assuming that management will set the protection level for the I_1 and I_2 net inventories sufficiently high so that backorders are incurred only in small quantities, the expected values of the net inventories will very closely approximate the expected values of the on-hand physical inventories. Accordingly, the mean value μ (where $\mu = \mu_1 + \mu_2$) of the combined ICP inventory, I, will be used to approximate the expected value of the physical inventory. The expected holding cost per period is, therefore, $h\mu$ and the expected yearly cost is $\frac{h\mu}{T}$.

The expected period cost of holding inventory at the repair facility is found by multiplying the repair holding cost (h_V) times the expected value of the physical inventory at the repair facility. Here it cannot be

assumed that backorders are incurred in small quantities. As a result of the derivation in subsection 2.5, a backorder or overage position is equally likely at the repair facility. Using the expected value of the physical inventory (expected overage) developed in subsection 2.5, the expected period cost of holding inventory at the repair facility is

$$\frac{h_V \, r \, \sigma_X}{\sqrt{\pi}} .$$

The expected annual cost of holding inventory at the repair facility is then

$$\frac{h_V \, r \, \sigma_X}{T \sqrt{\pi}} .$$

All the terms to be considered in the expected annual cost expression of the basic model have now been evaluated.

$$\text{Annual Cost} = \frac{1}{T} \left(A_P + A_R + h\mu + \frac{h_V \, r \, \sigma_X}{\sqrt{\pi}} \right) . \qquad (26)$$

Or, replacing μ with its equivalent (18) form, the annual cost can be expressed as follows:

$$\text{Annual Cost} = \frac{1}{T} \Big(A_P + A_R + h\, N(\alpha)\, \sigma_X \, ((L+1)^{\frac{1}{2}} (1-r) + (R + K^1 + 1)^{\frac{1}{2}} r) + \frac{h_V \, r \, \sigma_X}{\sqrt{\pi}} \Big) . \qquad (27)$$

Equation (27) relates the annual operating costs with the protection level α. As α is increased, the value of $N(\alpha)$ will increase, causing

the annual operating cost to increase. Since the basic model assumed a given review period length, the operating cost can only be varied by varying the protection level. Therefore, once the basic review period has been established, the decision-maker's final protection level choice will be governed by the associated annual operating costs he is willing to accept or, in the case of a budget constraint, the annual operating funds available.

3. EXTENSION OF THE BASIC MODEL

The structure of the basic model was based upon a given review period. Review and order action took place at the end of this unit period and ordered quantities were received at the beginning of the period. The following extension of the model to evaluate the effect of extending the ICP repaired-item review period and/or the purchased-item inventory review period will maintain the same relative time point relationships between these events within the extended periods. This requires that only those time periods that are integer multiples of the basic period and are evenly divisible into the original number of lead-time periods can be considered in the analysis.

Let T_1 be defined as an extended time period with length equal to an integer multiple of the basic time period such that $L \geq T_1 \geq 1$ and $\frac{L}{T_1}$ is an integer. Let T_2 be similarly defined as an integer multiple of the basic time period such that $R \geq T_2 \geq 1$ and $\frac{R}{T_2}$ is an integer. The demand over these extended periods is the sum of the independent basic period demands contained within the extended interval. It follows that the demand over the extended periods remains normally distributed. The mean and variance parameters of the extended period demand distributions can be obtained by scaling the corresponding basic model's I_1 and I_2 demand parameters (mean and variance) by the integer values of T_1 or T_2. The length of the purchase and repair order leadtimes in terms of the extended periods are represented by $\frac{L}{T_1}$ and $\frac{R}{T_2}$, respectively.

Using the preceding definitions, all the basic model equations in section 2 can be rederived in the same manner as before but in terms of the extended periods T_1 and T_2. A similar rederivation of the repair facility inventory distribution implies another restriction on the values that can be taken on by T_2. Recalling that the subsection 2.5 derivation assumed that the repair facility input was independent of the ICP repaired item demand during the previous period, it becomes necessary to further restrict T_2 to values less than the turn-around time K. Keeping in mind the restrictions on T_1 and T_2, the principle equations developed in section 2 can be rewritten as functions of T_1 and T_2 as follows.

(a) ICP Purchased Item Inventory (I_1)

$$\sigma_1^2 = (L + T_1)(1 - r)^2 \sigma_X^2 \tag{28}$$

$$\mu_1 = N(\alpha)(L + T_1)^{\frac{1}{2}} (1 - r) \sigma_X \tag{29}$$

$$\overset{*}{H}_1 = (1 - r)\left(N(\alpha)\sigma_X (L + T_1)^{\frac{1}{2}} + (L + T_1)\mu_X\right) \tag{30}$$

(b) ICP Repaired Item Inventory (I_2)

$$\sigma_2^2 = (R + K^1 T_2 + T_2) r^2 \sigma_X^2 \tag{31}$$

$$\mu_2 = N(\alpha)(R + K^1 T_2 + T_2)^{\frac{1}{2}} r \sigma_X \tag{32}$$

$$H_2 = r\left(N(\alpha)\,\sigma_X\,(R + K^1 T_2 + T_2)^{\frac{1}{2}} + (R + K^1 T_2 + T_2)\,\mu_X\right) \tag{33}$$

(c) Repair Facility Inventory (V)

$$\mu_V = 0 \tag{34}$$

$$\sigma_V^2 = 2\,T_2\,r^2\,\sigma_X^2 \tag{35}$$

$$E[\text{overage}] = \frac{(T_2)^{\frac{1}{2}}\,r\sigma_X}{\sqrt{\pi}} \tag{36}$$

$$K^1 = \left[\frac{\sigma_X}{(\pi\ T_2)^{1/2}\ \mu_X}\right] + 1 \tag{37}$$

(d) Expected Annual Operating Cost

$$\text{Annual Cost} = \frac{1}{T}\left(\frac{A_P}{T_1} + \frac{A_R}{T_2} + h\,N(\alpha)\,\sigma_X \cdot \left((L + T_1)^{\frac{1}{2}}(1 - r) + (R + K^1 T_2 + T_2)^{\frac{1}{2}}\,r\right) + \frac{h_V\,(T_2)^{\frac{1}{2}}\,r\,\sigma_X}{\sqrt{\pi}}\right) \tag{38}$$

The values of T_1 and T_2 that minimize the annual operating cost expression (38) for a given protection level (α) can be determined in the following manner:

(1) Hold T_2 constant at a value of one. Solve (38) for all integer values of T_1 permitted by the model. Select that value of T_1 which minimizes (38).

(2) Hold T_1 at the value determined above. Solve (38) for all integer values of T_2 permitted by the model. Select that value of T_2 which minimizes (38).

Using the above values of T_1 and T_2, the corresponding high limits H_1 and H_2 can be calculated from (30) and (33).

4. EXAMPLE

The following numerical example is used to indicate the nature of the solutions given by the model. The parameter values chosen are as follows:

$$
\begin{aligned}
T &= 1/12 \text{ year} \\
\mu_X &= 40 \\
\sigma_X &= 4 \\
r &= .9 \\
L &= 9 \text{ months} \\
R &= 4 \text{ months} \\
K &= 5 \text{ months} \\
A_P &= \$200 \\
A_R &= \$100 \\
h &= \$20 \text{ per item - month} \\
h_V &= \$10 \text{ per item - month} \\
\alpha &= .95 \text{ or } .99
\end{aligned}
$$

With these values, the following results were obtained:

95% Protection (α = .95)	99% Protection (α = .99)
$N(\alpha) = 1.65$	$N(\alpha) = 2.326$
$K^1 = 1$	$K^1 = 1$
$T_1 = 9$ (months)	$T_1 = 9$ (months)
$T_2 = 1$ (month)	$T_2 = 1$ (month)
$H_1 = 74.8$	$H_1 = 75.9$
$H_2 = 229.6$	$H_2 = 236.5$
Annual Cost = \$5,873	Annual Cost = \$7,580

The costs per year of operating the inventory system under the basic model ($T_1 = 1$, $T_2 = 1$) were calculated to be \$7,835 and \$9,471 for the 95% and 99% protection levels, respectively. The combination of $T_1 = 9$ and $T_2 = 1$ produced the minimum annual costs under both the selected protection levels as shown above. The other combinations permitted by the model, i.e., $T_1 = 1, 3, 9$, $T_2 = 1, 2, 4$, generated higher annual costs.

The model is highly sensitive to the standard deviation of demand, σ_X . For example, if σ_X was actually 8 vice 4 units as used in the above example, the 99% protection level would be reduced to 87% (holding the cost and other operating variables constant). Retaining the same review period combination $T_1 = 9$ and $T_2 = 1$, this sensitivity can be illustrated in another manner. If σ_X were raised from 4 to 8 units, the same 99% protection level computation would cause the expected annual operating cost to almost double from \$7,580 to \$14,891 .

The sensitivity of the model to the repair and purchase order leadtimes, R and L, depends upon the values of σ_X and r. As σ_X increases, the model becomes more sensitive to the leadtimes which are scaled by r and (1 - r), respectively. For example, if the repair order leadtime for the 99% protection example was reduced from 4 to 2 months, the expected annual operating cost would be reduced only \$903 from \$7,580 to \$6,677. However, if σ_X were 8 vice 4 units, this same reduction in leadtime would cause the expected annual operating cost to be reduced \$3,004 from \$14,891 to \$11,887.

The selection of the protection levels (95% and 99%) for the examples was arbitrary. A family of solutions could be obtained for various protection levels and presented to management for the selection of that protection level that best met the funding constraint for the particular item in question.

The fact that the example's minimum cost was found at $T_1 = 9$ points up the limitation of the extended model. This value represents an end-point solution in that this is the maximum value that can be taken on by T_1 within the extended model structure. The possibility that additional savings might be realized by extending the review period length even further remains unresolved. Looking in the other direction, it is possible to evaluate the effects of period lengths smaller than the example's basic unit period T by equating T to a smaller period, i.e., a week vice a month, and using correspondingly smaller demand data.

Nevertheless, the relatively few discrete points at which annual operating costs can be evaluated remains a limiting characteristic of the model.

5. CONCLUSIONS AND RECOMMENDATIONS

As a result of the assumptions and the evaluation restrictions of section 3, the model's scope of application is necessarily confined to particular cases. Within this scope, the equations provide a means of obtaining an understanding of the interactions of the principle parameters and how they affect the inventory positions and costs associated with operating a system of this nature. The model's equations indicate the sensitivity of the reparable-item system inventories to order leadtimes, return rates, mean demand, and protection levels. More significantly, the equations highlight the high sensitivity of the system to the demand variance parameter.

Possible courses of action to improve the model of the periodic review system addressed by this paper include the following:

(1) Define an acceptable method and/or criterion whereby material from either one of the two ICP inventories treated by this paper could be used to satisfy demand experienced by the other. This could be permitted in the present basic model if one accepts the rationale that the ordering rules are based upon "paper" inventory positions and that the physical inventories could be co-mingled. This would permit the use of the basic model's combined inventory variance term (19) in an overall protection level computation. However, under the present structure, the covariance between the two inventories would be difficult to evaluate in the extended model cases due to the changing dependence relationships between repaired and purchased item demand over varying period lengths.

(2) Modify the periodic review structure in such a manner that the integer restrictions that confine the present model could be relaxed. This would permit finer resolution of the least cost review period combination and more exact treatment of the expected delay time K^1 at the repair facility.

This paper looked upon the reparable item system as two subsystems. "Optimizing" this type of structure eventually leads to combining sub-optimized parts vice truly optimizing the whole system. However, if one subsystem is significantly larger than the other, the shortcomings normally associated with suboptimization should be considerably less than if both subsystems are of equal stature. As a result of high recovery rates (estimated at 90 - 95%) being experienced on reparable items, the repaired item subsystem addressed by this paper would be proportionally larger (in terms of items controlled) than the purchased item subsystem. Under these conditions, the savings that would be realized by optimizing the total system might well be less than the costs involved in developing and operating under the more complex ordering rules required by total system optimizing techniques. Clearly, a trade-off does exist and a study to better establish this relationship would serve to channel future modeling efforts into the most promising direction.

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APPENDIX A

The portion of the total demand being satisfied from the purchased-item inventory I_1 is $(1 - r)X$, where X is a normally distributed random variable with parameters μ_X and σ_X^2. As shown in reference [7], the probability density function for $(1 - r)X$ is given by

$$F_{(1-r)X}(y) = \frac{1}{1-r} f_X\left(\frac{y}{1-r}\right) . \tag{A - 1}$$

From (A - 1), it follows that $(1 - r)X$ is a normally distributed random variable with parameters:

$$\text{mean} = \mu_X (1 - r) ,$$

$$\text{variance} = \sigma_X^2 (1 - r)^2 .$$

It was shown in subsection 2.2 that

$$I_1 = H_1 - \sum_{i=1}^{L+1} (1 - r) X_i . \tag{A - 2}$$

Let

$$Z = \sum_{i=1}^{L+1} (1 - r) X_i .$$

Due to the assumed independence of the X_i's, Z is also normally distributed with parameters:

$$\begin{aligned} \mu_Z &= (L + 1) \mu_X (1 - r) , \\ \sigma_Z^2 &= (L + 1) \sigma_X^2 (1 - r)^2 . \end{aligned} \tag{A - 3}$$

The distribution function $F(x)$ of the random variable I_1 is determined as follows:

$$\begin{aligned} F_{I_1}(x) &= P[I_1 \le x] \\ &= P[H_1 - Z \le x] \\ &= P[Z \ge H_1 - x] \\ &= 1 - F_Z(H_1 - x) \\ &= 1 - \int_{-\infty}^{H_1 - x} f_Z(z)\, dz \quad . \end{aligned} \tag{A - 4}$$

In terms of a standard normal density function $\phi(y)$, equation (A - 4) can be written as

$$F_{I_1}(x) = 1 - \int_{-\infty}^{\frac{H_1 - x - \mu_Z}{\sigma_Z}} \phi(y)\, dy \quad .$$

Therefore,

$$\begin{aligned} f_{I_1}(x) &= \frac{1}{\sigma_Z}\, \phi\left(\frac{H_1 - x - \mu_Z}{\sigma_Z}\right) \\ &= \frac{1}{\sigma_Z}\, \phi\left(\frac{x - H_1 + \mu_Z}{\sigma_Z}\right) \\ &= \frac{1}{\sqrt{2\pi}\, \sigma_Z}\, e^{-\frac{1}{2\sigma_Z^2}[x - (H_1 - \mu_Z)]^2} \quad . \end{aligned}$$

It follows that I_1 is normally distributed with the following parameters (replacing μ_Z and σ_Z with their equivalent (A - 3) forms):

$$\mu_1 = H_1 - (L + 1)(1 - r)\mu_X ,$$

$$\sigma_1^2 = (L + 1)(1 - r)^2 \sigma_X^2 .$$

APPENDIX B

The covariance of I_1 and I_2, $\mathrm{Cov}(I_1, I_2)$, is derived as follows:

$$\mathrm{Cov}(I_1, I_2) = E[I_1 I_2] - E[I_1] E[I_2] \quad , \qquad \text{(B-1)}$$

where $E[X] \equiv$ expected value of X .

From equations (3) and (11), we can write

$$E[I_1 I_2] = E\Big[\Big(H_1 - \sum_{i=1}^{L+1} (1-r) X_i\Big) \cdot \Big(H_2 - \sum_{i=1}^{R+K^1+1} r X_i\Big)\Big]$$

$$= H_1 H_2 - H_2 (L+1)(1-r)\mu_X - H_1 (R + K^1 + 1) r \mu_X$$

$$+ (1-r) r \; E\Big[\sum_{i=1}^{L+1} X_i \sum_{i=1}^{R+K^1+1} X_i\Big] \; .$$

Based on the condition that $L \geq R + K^1$,

$$E\Big[\sum_{i=1}^{L+1} X_i \sum_{i=1}^{R+K^1+1} X_i\Big] = E\Big[\sum_{i=1}^{R+K^1+1} X_i^2\Big] + E\Big[\sum_{i<j}^{R+K^1+1}\sum 2 X_i X_j\Big]$$

$$+ E\Big[\sum_{i=1}^{R+K^1+1} \sum_{j=R+K^1+2}^{L+1} X_i X_j\Big]$$

$$= (R + K^1 + 1)(\sigma_X^2 + \mu_X^2) + (R + K^1) \cdot (R + K^1 - 1)\mu_X^2 + (R + K^1 + 1)(L - R - K^1)\mu_X^2$$

$$= (R + K^1 + 1)(\sigma_X^2 + (L+1)\mu_X^2) \; .$$

Therefore,

$$E[I_1 I_2] = H_1 H_2 - H_2 (L + 1)(1 - r)\mu_X - H_1 (R + K^1 + 1) r \mu_X$$

$$+ (1 - r) r (R + K^1 + 1)(\sigma_X^2 + (L + 1)\mu_X^2) . \qquad (B - 2)$$

Now,

$$E[I_1] E[I_2] = \mu_1 \mu_2 = (H_1 - (L + 1)(1 - r)\mu_X)$$

$$\cdot (H_2 - (R + K^1 + 1) r \mu_X) . \qquad (B - 3)$$

Putting the (B - 2) and (B - 3) forms of $E[I_1 I_2]$ and $E[I_1] E[I_2]$ back into equation (B - 1) and clearing terms gives

$$\mathrm{Cov}(I_1, I_2) = r(1 - r)(R + K^1 + 1)\sigma_X^2 . \qquad (B - 4)$$

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13. ABSTRACT

The purpose of this thesis is to structure a periodic review model for a reparable item system experiencing normally distributed random demand. The items are issued upon demand, repaired (if possible) after use or failure, and subsequently reissued. The model addresses the requirement of system replenishment as items are lost or determined to be beyond economical repair. The inventory of ready-for-issue items is treated as two separate inventories, i.e., an inventory of those items received directly from the manufacturer and an inventory of those items received from the repair facility. Annual system cost expressions are developed as a function of the desired protection level and length of review period.

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14. KEY WORDS	LINK A		LINK B		LINK C	
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